Math-109: Pre-Calculus Algebra
Section: 8
Solutions to Midterm Exam 3

Name: ____________________
ULID: ____________________

Please write complete step by step solutions (whenever possible) to the problems below. Show your work

1. (5 points) Find a polynomial of minimum degree that has the given zeros. Get rid of the imaginary i's in the expression.
   Zeros: 1,3, $2-i$, $2+i$.

   Proof.

$$\begin{aligned} P(x) &= (x-1)(x-3)(x-(2-i))(x-(2+i)) \\ &= (x-1)(x-3)(x-2+i)(x-2-i) \\ &= (x-1)(x-3)((x-2)^2-i^2) \\ &= (x-1)(x-3)((x-2)^2+1) \\ &= (x-1)(x-3)(x^2-4x+5) \end{aligned}$$

□

2. (5 points) Factor the following polynomial as a product of **linear** factors.
   $P(x) = (x^2-1)(x^4-16)$.

   Proof. We have

$$\begin{aligned} P(x) &= (x^2-1^2)((x^2)^2-4^2) \\ &= (x+1)(x-1)(x^2+4)(x^2-4) \\ &= (x+1)(x-1)(x^2-(2i)^2)(x^2-2^2) \\ &= (x+1)(x-1)(x+2i)(x-2i)(x+2)(x-2) \end{aligned}$$

□

3. (9 points) Solve the following quadratic equations. Find all solutions including complex zeros.

   (a) $x^2+3x+5=0$

   Proof.

$$\begin{aligned} x &= \frac{-3 \pm \sqrt{3^2 - 4\cdot 1 \cdot 5}}{2\cdot 1} \\ &= \frac{-3 \pm \sqrt{9-20}}{2} \\ &= \frac{-3 \pm \sqrt{-11}}{2} \\ &= \frac{-3 \pm \sqrt{11}i}{2} \end{aligned}$$

□

(b) $(x+5)^2 = -9$

Proof. Taking square roots on both sides we get

$$\begin{aligned}(x+5) &= \pm\sqrt{-9}\\ x+5 &= \pm\sqrt{9}i\\ x+5 &= \pm 3i\\ x &= -5 \pm 3i\end{aligned}$$

□

(c) $2x^2 + 7x + 4 = 0$

Proof.

$$\begin{aligned}x &= \frac{-7 \pm \sqrt{7^2 - 4\cdot 2\cdot 4}}{2\cdot 2}\\ &= \frac{-7 \pm \sqrt{49-32}}{4}\\ &= \frac{-7 \pm \sqrt{-17}}{2}\\ &= \frac{-7 \pm \sqrt{17}i}{2}\end{aligned}$$

□

4. (6 points) Find the domain of the following rational functions.

(a) $\frac{2x+18}{4x^2-36}$

Proof. We have to exclude the numbers for which the denominator is zero. Hence we have to solve

$$\begin{aligned}4x^2 - 36 &= 0\\ 4(x^2-9) &= 0\\ x^2 - 9 &= 0\\ x^2 &= 9\\ x &= \pm 3\end{aligned}$$

Thus, domain is $(-\infty, -3) \cup (-3, 3) \cup (3, \infty)$. □

(b) $\frac{x^3+x^2+1}{2(x^2-3x-4)}$.

Proof. When is the denominator zero? We have to solve

$$\begin{aligned}2(x^2 - 3x - 4) &= 0\\ x^2 - 3x - 4 &= 0\\ (x-4)(x+1) &= 0\end{aligned}$$

Hence, $x = 4$ or $x = -1$. Thus, the domain is $(-\infty, -1) \cup (-1, 4) \cup (4, \infty)$ □

(c) $\frac{x^5+x^4+x^3+x^2+x+1}{x^2+1}$

Proof. We have to exclude the real numbers x for which the denominator equals 0. We solve

$$\begin{aligned} x^2 + 1 &= 0 \\ x^2 &= -1 \\ x &= \pm i \end{aligned}$$

Since, the above equation does not have real solutions, the domain is $(-\infty, \infty)$. □

5. (10 points) For the rational function $f(x) = \frac{3x+6}{x-2}$, find:

(a) Find any x-intercepts.

Proof. We have to solve

$$\begin{aligned} f(x) &= 0 \\ \frac{3x+6}{x-2} &= 0 \\ 3x + 6 &= 0 \\ 3x &= -6 \\ x &= -2 \end{aligned}$$

Therefore, the x-intercept is $(-2, 0)$.

□

(b) Find any y-intercepts.

Proof. We need to plug $x = 0$ into $f(x)$. We get

$$\begin{aligned} f(0) &= \frac{3 \cdot 0 + 6}{0 - 2} \\ &= \frac{0 + 6}{-2} \\ &= \frac{6}{2} \\ &= 3 \end{aligned}$$

Thus, the y-intercept is $(0, 3)$. □

(c) Find the equation of the vertical asymptote.

Proof. Note that there are no common factors in the numerator and denominator. Hence, the vertical asymptotes are just the zeros of the denominator. The denominator is zero when $x = 2$. Thus, the vertical asymptote is the line $x = 2$.

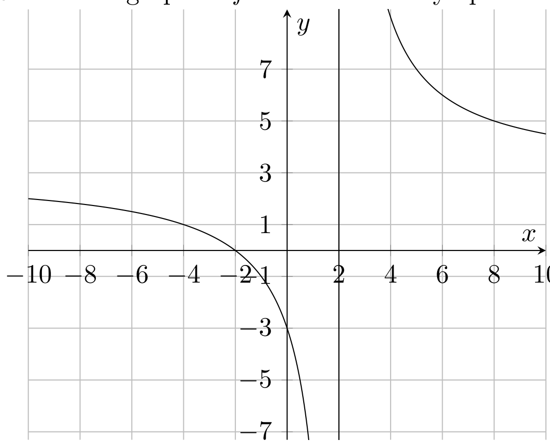
□

(d) Find the equation of the horizontal asymptote.

Proof. Since the numerator and the denominator have the same degree, the horizontal asymptote is $y = \frac{a_n}{b_m} = \frac{3}{1} = 3$.

□

(e) Sketch the graph of f and label all asymptotes and intercepts.



6. (8 points) Solve the given system of linear equations.

$$3x + 2y = 1 \qquad (i.)$$
$$5x + 7y = 9 \qquad (ii.)$$

Proof. Multiplying eq i. by -5 and eq ii. by 3 we have

$$-15x - 10y = -5 \qquad (iii.)$$
$$15x + 21y = 27 \qquad (iv.)$$

Adding equation (iii.) and (iv.) we get

$$11y = 22$$
$$y = 2$$

Plugging this value of y into equation i. we get

$$3x + 2 \cdot 2 = 1$$
$$3x = 1 - 4$$
$$3x = -3$$
$$x = -1$$

Therfore, $x = -1$ and $y = 2$. □

$$\frac{1}{2}x - \frac{1}{4}y = 2 \qquad i.$$
$$4x - 3y = 10 \qquad ii.$$

Proof. Multplying equation i. by -8, we get

$$-4x + 2y = -16 \qquad iii.$$

Adding equations ii. and iii. we get

$$-y = 6$$
$$y = -6$$

Plugging $y = -6$ into equation ii. we get

$$\begin{aligned} 4x - 3 \cdot (-6) &= 10 \\ 4x &= 10 + 18 \\ x &= 7 \end{aligned}$$

□

$$\begin{aligned} 2x + y &= 3 \qquad \text{i.} \\ 4x + 2y &= 4 \qquad \text{ii.} \end{aligned}$$

Proof. Multiplying equation i. by -2 we get

$$-4x - 2y = -6 \qquad \text{iii.}$$

Adding equations iii. and ii. we get

$$0 = -2$$

This is a contradiction. Thus, there is no solution. □

$$\begin{aligned} x + 2y &= 1 \qquad \text{i.} \\ 2x + 4y &= 2 \qquad \text{ii.} \end{aligned}$$

Proof. Multiplying equation i. by -2 we get

$$-2x - 4y = -2 \qquad \text{iii.}$$

Adding equations iii. and ii. we get

$$0 = 0$$

This is always true. Hence, there are infinitely many solutions and the solution is the line $x+2y = 1$. □

7. (12 points) Write the augmented matrix if given the system of linear equations, and the system of equations if given the augment matrix.

(a)

$$\begin{aligned} 4x + 2y - 3z &= 0 \\ 2x + y + z &= -9 \\ x + y - 4z &= 7 \end{aligned}$$

Proof.

$$\left(\begin{array}{ccc|c} 4 & 2 & -3 & 0 \\ 2 & 1 & 1 & -9 \\ 1 & 2 & -4 & 7 \end{array}\right)$$

□

(b)

$$\begin{aligned} x + 2y - 3z &= -1 \\ y + z &= 1 \\ x - 4z &= -2 \end{aligned}$$

Proof.

$$\left(\begin{array}{ccc|c} 1 & 2 & -3 & -1 \\ 0 & 1 & 1 & 1 \\ 1 & 0 & -4 & -2 \end{array}\right)$$

□

(c)

$$\begin{aligned} 3z + 11x + 2y &= 0 \\ 44x + y + z &= -6 \end{aligned}$$

Proof.

$$\left(\begin{array}{ccc|c} 11 & 2 & 3 & 0 \\ 44 & 1 & 1 & -6 \end{array}\right)$$

□

(d)

$$\left(\begin{array}{cccc|c} 0 & 1 & 1 & -1 & 7 \\ 2 & 2 & 3 & 0 & 1 \\ 10 & 1 & 2 & 4 & -9 \end{array}\right)$$

Proof.

$$\begin{aligned} x + y - z &= 7 \\ 2w + 2x + 3y &= 1 \\ 10w + x + 2y + 4z &= -9 \end{aligned}$$

□

8. (5 points) Solve the following system using Gauss-Jordan elimination with back substitution.

$$\begin{aligned} x + y - z &= 0 \\ 2x + y + z &= 1 \\ 2x - y + 3z &= -1 \end{aligned}$$

Proof.

$$\left(\begin{array}{ccc|c} 1 & 1 & -1 & 0 \\ 2 & 1 & 1 & 1 \\ 2 & -1 & 3 & -1 \end{array}\right) \xrightarrow{\text{(-2)R1+R2}\to\text{R2}} \left(\begin{array}{ccc|c} 1 & 1 & -1 & 0 \\ 0 & -1 & 3 & 1 \\ 2 & -1 & 3 & -1 \end{array}\right) \xrightarrow{\text{(-2)R1+R2}\to\text{R2}}$$

$$\left(\begin{array}{ccc|c} 1 & 1 & -1 & 0 \\ 0 & -1 & 3 & 1 \\ 0 & -3 & 5 & -1 \end{array}\right) \xrightarrow{\text{(-1)R2}\to\text{R2}} \left(\begin{array}{ccc|c} 1 & 1 & -1 & 0 \\ 0 & 1 & -3 & -1 \\ 0 & -3 & 5 & -1 \end{array}\right) \xrightarrow{\text{3R2+R3}\to\text{R3}}$$

$$\left(\begin{array}{ccc|c} 1 & 1 & -1 & 0 \\ 0 & 1 & -3 & -1 \\ 0 & 0 & -4 & -4 \end{array}\right) \xrightarrow{\text{-1/4 R3}\to\text{R3}} \left(\begin{array}{ccc|c} 1 & 1 & -1 & 0 \\ 0 & 1 & -3 & -1 \\ 0 & 0 & 1 & 1 \end{array}\right)$$

Converting the last matrix to system of equations we have

$$\begin{aligned} x + y - z &= 0 \\ y - 3z &= -1 \\ z &= 1 \end{aligned}$$

Plugging $z = 1$ into the second equation:

$$\begin{aligned} y - 3 \cdot 1 &= -1 \\ y &= -1 + 3 \\ y &= 2 \end{aligned}$$

Pluggin $y = 2$ and $z = 1$ into the first equation we have

$$\begin{aligned} x + 2 - 1 &= 0 \\ x + 1 &= 0 \\ x &= -1 \end{aligned}$$

□